Math 53, Discussions 116 and 118 **2022-10-21**

Double and triple integrals

Answers

Questions

Question 1. A lamina of constant density ρ occupies the half-disk $x^2 + y^2 \leq 1$, $x \geq 0$. Find its center of mass. Feel free to use geometry where applicable.

Question 2. Let

$$f(x) = \int_x^1 \cos(t^2)\,\mathrm{d}t.$$

Note that this is a function of x, *not* of t. Find the average value of f on the interval $[0,1]$.

(The average value of a function over some region can be computed by integrating the function over that region, and then dividing by the size of the region.)

Question 3. Rewrite the following integrals in the five other integration orders.

(a) $\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$

(b) $\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$

Question 4. Show that

$$\int_{-1}^1 \int_0^2 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \arctan(x\sqrt{y}\cos(z))e^{-zx^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 0$$

by changing the order of integration so that x is done first.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. Since the mass is distributed symmetrically across the x-axis, we know that $\bar{y} = 0$. We have

$$\bar{x} = \frac{\iint_D x\rho \, dA}{\iint_D \rho \, dA} = \frac{\iint_D x \, dA}{\iint_D dA}$$

where D denotes the half-disk under consideration. The denominator is just the area of D, which is $\pi/2$, and the numerator can be computed in polar:

$$\begin{aligned} \iint_D x \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^1 (r\cos\theta) r \, dr \, d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{1}{3}(1)^3 \cos\theta \, d\theta \\ &= 2/3. \end{aligned}$$

Hence $(\bar{x}, \bar{y}) = (\frac{4}{3\pi}, 0)$.

Question 2. The average value of f is

$$\frac{\int_0^1 f(x) \, dx}{1-0} = \int_0^1 \int_x^1 \cos(t^2) \, dt \, dx.$$

Changing the order of integration gives

$$\begin{aligned} \int_0^1 \int_x^1 \cos(t^2) \, dt \, dx &= \int_0^1 \int_0^t \cos(t^2) \, dx \, dt \\ &= \int_0^1 t\cos(t^2) \, dt \\ &= \frac{1}{2}\sin(1). \end{aligned}$$

Question 3. Here are the other ways of writing the integral in (a):

- $\int_0^1 \int_z^1 \int_y^1 f(x,y,z) \, dx \, dy \, dz$
- $\int_0^1 \int_0^y \int_y^1 f(x,y,z) \, dx \, dz \, dy$
- $\int_0^1 \int_0^x \int_0^y f(x,y,z) \, dz \, dy \, dz$
- $\int_0^1 \int_z^1 \int_z^x f(x,y,z) \, dy \, dx \, dz$
- $\int_0^1 \int_0^x \int_z^x f(x,y,z) \, dy \, dz \, dx$

For the integral in (b), if you do dz first (i.e. $dz \, dx \, dy$ or $dz \, dy \, dx$) you will have to split the integral into two. For example, for $dz \, dx \, dy$,

$$\int_0^1 \int_y^1 \int_x^1 f(x,y,z) \, dz \, dx \, dy + \int_0^1 \int_0^y \int_y^1 f(x,y,z) \, dz \, dx \, dy.$$

The other integration orders are more straightforward and I'll omit them.

Question 4. The region of integration is the solid

$$\begin{aligned} -\sqrt{1-x^2} \le z &\le \sqrt{1-x^2} \\ 0 \le y &\le 2 \\ -1 \le x &\le 1. \end{aligned}$$

Take a moment to sketch this region: it is a solid cylinder. If we change to e.g. the $dx \, dy \, dz$ integration order, we get the bounds

$$\int_{-1}^1 \int_0^2 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \arctan(x\sqrt{y}\cos(z)) e^{-zx^2} \, dx \, dy \, dz.$$

But in fact, the innermost integral

$$\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \arctan(x\sqrt{y}\cos(z)) e^{-zx^2} \, dx$$

is equal to zero, because the integrand is an odd function of x (check this). So the whole integral is equal to zero.